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LMH Physics Open Book Exercise for visiting students

Please attempt all the questions in both sections A and B

The use of calculators is permitted

There is no time limit, however, it is expected that candidates should be able to complete the exercise within approximately 3 hours

When completed, please scan and save your answers as a PDF file and upload as supporting documentation using the following file name:
'your surname, your name_Physics Open Book Exercise.pdf'

Section A

1. A small, smooth ball of mass $3m$ is attached to one end of a light rod of length l which is freely pivoted at the other end. The ball is released from rest with the rod at an angle θ to the vertical. As the rod swings through the vertical position, the ball collides elastically head-on with a second small, smooth ball of mass m which is sitting at rest on a smooth table. Find an expression for the speed with which the lighter ball moves off as a function of the release angle θ. [5]

2. In a very simple microwave oven, electrons are accelerated through a potential difference V and then introduced into a region with no electric field and a uniform magnetic field, $\mathbf{B}$, which is perpendicular to their direction of motion. Given that the force experienced by the electrons is given by $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, write down the equations of motion of the electrons. Show that the electrons move in a circle and that the frequency of their orbits is independent of V. The electrons emit radiation at a frequency equal to their orbital rotation frequency. If this frequency is 2.5 GHz, and the accelerating potential V is 1kV, calculate the value of $\mathbf{B}$ and the radius of the circular motion. [5]

3. A rocket of mass m_0 and carrying fuel of additional mass M is at rest in deep space. It propels itself by ejecting fuel at a constant rate and at a non-relativistic speed v_0 with respect to the rocket. Show that when starting from rest, the speed of the rocket v_f once all the fuel has been ejected is given by

$$v_f = v_0 \ln\left(1 + \frac{M}{m_0}\right).$$

[5]

4. A spaceship of rest mass 10^5 kg is travelling at a constant speed v to the nearest star which is 4 light years away. Calculate the kinetic energy of the spacecraft if the journey takes 5 years from the point of view of the astronauts on the spaceship. [6]

5. A simple harmonic oscillator has a mass m, spring constant k, and is constrained to move in the x direction. Evaluate both the Lagrangian, L, and the Hamiltonian, H. Show that the total energy is conserved. [4]

6. State the parallel axis and perpendicular axis theorems of moment of inertia. [2]

A weightlifter's dumbell can be modelled as a thin rod of mass m_0 and length l with a thin disk of radius a and mass M at each end. Calculate the moment of inertia of the dumbell through the centre of the rod about an axis perpendicular to the rod. (You may assume that the moment of inertia of a disk of radius a and mass m about an axis through its centre and perpendicular to the plane of the disk is $\frac{1}{2}ma^2$). [5]

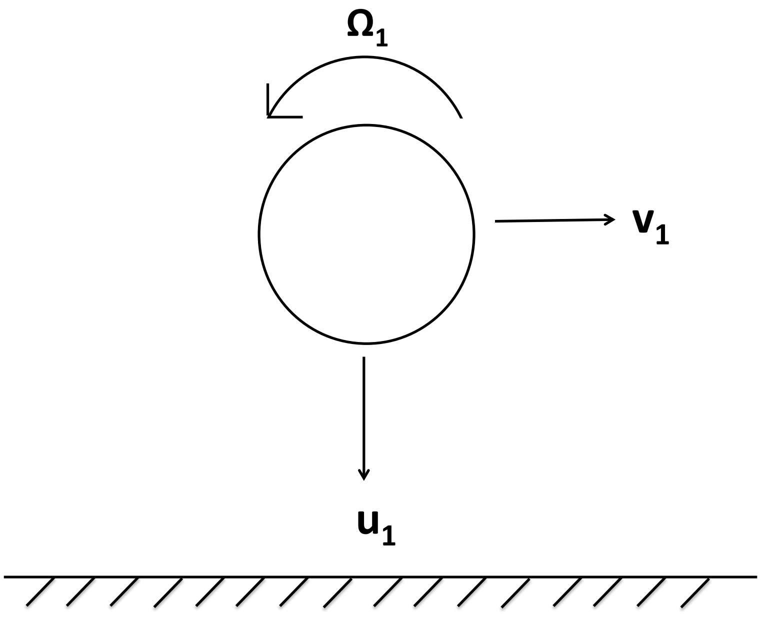
7. A projectile of mass m is launched at time $t = 0$ from ground level with a speed v_0 at an angle θ to the horizontal. The projectile is subject to a drag force, $\mathbf{F}_{\mathrm{drag}} = -\alpha\mathbf{v}$, where $\mathbf{v}$ is the instantaneous velocity. Find equations for the horizontal and vertical components of the projectile's velocity as a function of time. Sketch the two components of velocity as a function of time in the cases (i) $\alpha = 0$ and (ii) $\alpha > 0$. [8]

Section B

8. Show that the moment of inertia of a disk of radius a and mass m about an axis through its centre and perpendicular to the plane of the disk is $\frac{1}{2}ma^2$. [3]

By dividing a sphere into thin disks, or otherwise, calculate the moment of inertia of a uniform sphere, of mass M and radius a, about an axis through its centre. [7]

A spherical superball is perfectly elastic, incompressible and rough. In the figure below a superball of radius a is spinning with an angular velocity Ω_1 as shown. The superball hits a rough, horizontal surface while travelling with components of velocity u_1 normal to the surface and v_1 parallel to the surface.



Sketch and explain the forces acting on the superball at the moment of contact with the surface. [3]

Find the linear and angular velocities u_2, v_2 and Ω_2 with which it rebounds. [7]

9. (a) Write down expressions for the momentum p and the total energy E of a relativistic particle, and use them to verify that $E^2 - p^2c^2$ is an invariant. [3]

A particle of mass m travelling at speed v_1 collides head-on with a particle of mass $2m$ travelling at speed $v_2 = c/3$. They merge to form a single particle of mass M at rest in the same frame. Find expressions for v_1 and M. Comment on why M is not equal to $3m$. [6]

(b) Write down the Lorentz transforms for energy and momentum. Consider a photon of energy E' travelling in a frame S' which is moving at speed $v = \beta c$ with respect to a frame S. Show that the relativistic Doppler factor, D, is given by

$$D = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$$

where $D = E/E'$ is the ratio of the photon energies. [3]

In the case where the relative speed is close to the speed of light, c, i.e. $\beta = (1-\epsilon)$, where ϵ is small, show that the Doppler factor can be written as

$$D = \left(4\gamma^2 - 1\right)^{\frac{1}{2}}$$

where γ is the Lorentz factor. [2]

(c) The propagation of extremely high energy photons through the universe is limited by a reaction with the cosmic microwave background radiation in which electron-positron pairs are formed:

$$\gamma_1 + \gamma_2 \rightarrow e^+ + e^-.$$

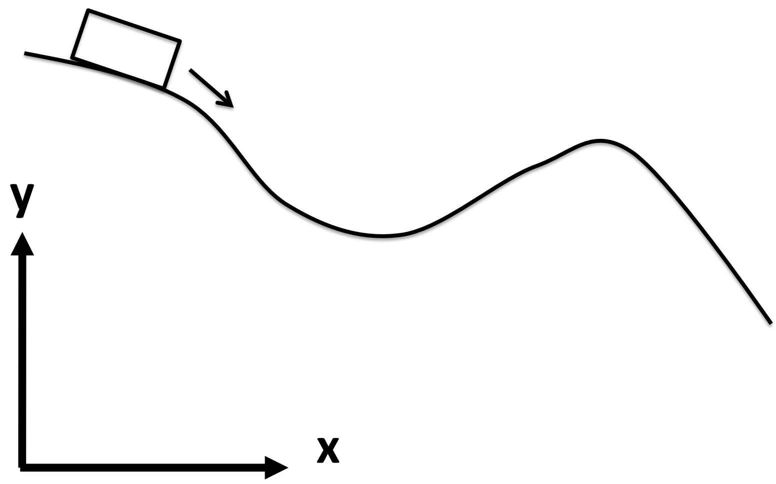
Assuming that the background photon γ_2 has an energy of 10^{-3} eV, calculate the minimum energy that the other photon γ_1 must have in order for the above reaction to take place. [4]

What is the velocity of the resulting electron-positron pair in the laboratory frame? [2]

10. (a) Write down the Euler-Lagrange equations, defining all the terms and explain how these equations can be used to derive the equations of motion for a system. Explain what is meant by a cyclic co-ordinate. [4]

(b) A block of mass m slides down a smooth plane which is at rest and fixed at an angle θ to a horizontal surface. By resolving forces, calculate the acceleration of the block in the horizontal, x direction. [4]

(c) In the figure below, a cart of mass m slides along a frictionless roller coaster track whose vertical height as a function of horizontal distance is $y(x)$.



Using x as the co-ordinate, determine the conjugate momentum, p_x, and use the Euler-Lagrange equations to show that the equation of motion can be written as:

$$\ddot{x}\left[1+\left(\frac{dy}{dx}\right)^2\right] = -g\frac{dy}{dx} - \dot{x}^2\frac{dy}{dx}\frac{d^2y}{dx^2}\,.$$

[8]

In the case that $\frac{d^2y}{dx^2} \to 0$, calculate and comment on the subsequent motion in the x direction. Compare your answer to that found in part (b). [4]

11. A spacecraft of mass m is approaching Jupiter at an initial relative speed of v_0 when far from the planet, and with an impact parameter b. Show that the energy of the orbit can be written as

$$E = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2} - \frac{GmM_J}{r}$$

where M_{J} is the mass of Jupiter, and explain each term in the equation. [4]

Calculate an expression for the distance of closest approach, r_{min}, in terms of v_0, b, M_{J} and G. [4]

Show that the angle of deflection of the spacecraft's path relative to Jupiter, ϕ, is given by

$$\cot(\phi/2) = bv_0^2/GM_{\mathrm{J}}.$$

[6]

If $v_0 = 10\,\mathrm{km\,s^{-1}}$ and $M_{\mathrm{J}} = 2 \times 10^{27}\,\mathrm{kg}$, calculate the value of b such that the spacecraft trajectory is deflected by 90 degrees. [2]

The orbital velocity of Jupiter relative to the Sun is $13\,\mathrm{km\,s^{-1}}$. If after being deflected the spacecraft is travelling in the same direction as Jupiter, what are the velocities of the spacecraft relative to the Sun before and after the encounter? [4]